Carleson type measures for Fock–Sobolev spaces

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Abstract. We describe the (p, q) Fock–Carleson measures for weighted Fock–Sobolev spaces in terms of the objects (s, t)-Berezin transforms, averaging functions, and averaging sequences on the complex space $\mathbb{C}^n$. The main results show that while these objects may have growth not faster than polynomials to induce the (p, q) measures for $q \geq p$, they should be of $L^{p/(p-q)}$ integrable against a weight of polynomial growth for $q < p$. As an application, we characterize the bounded and compact weighted composition operators on the Fock–Sobolev spaces in terms of certain Berezin type integral transforms on $\mathbb{C}^n$. We also obtained estimation results for the norms and essential norms of the operators in terms of the integral transforms. The results obtained unify and extend a number of other results in the area.

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1. Introduction

The classical weighted Fock space $\mathcal{F}_\alpha^p$ consists of entire functions f on $\mathbb{C}^n$ for which

$$\|f\|_p^p = \left(\frac{\alpha p}{2\pi}\right)^n \int_{\mathbb{C}^n} |f(z)|^p e^{-\frac{\alpha p}{2}|z|^2} dV(z) < \infty$$

where dV denotes the usual Lebesgue measure on $\mathbb{C}^n$, $0 < p < \infty$, and α is a positive parameter. For $p = \infty$, the corresponding space consists of all such f's for which

$$\|f\|_\infty = \sup_{z \in \mathbb{C}^n} |f(z)| e^{-\frac{\alpha}{2}|z|^2} < \infty.$$

The space $\mathcal{F}_\alpha^2$, in particular, is a reproducing kernel Hilbert space with kernel and normalized reproducing kernel functions respectively given by $K_w(z) =$

$e^{\alpha\langle z,w\rangle}$ and $k_w(z) = e^{\alpha\langle z,w\rangle - \alpha|w|^2/2}$ where

$$\langle z, w\rangle = \sum_{j=1}^{n} z_j\overline{w_j}, \quad |z| = \sqrt{\langle z, z\rangle}, \; w = (w_j), z = (z_j) \in \mathbb{C}^n.$$

For an n-tuple $\beta = (\beta_1, \beta_2, ..., \beta_n)$ of nonnegative integers we also write $\partial^\beta = \partial_1^{\beta_1}...\partial_n^{\beta_n}$ where ∂_j denotes partial differentiation with respect to the j-th component. For any non-negative integer m and $0 < p \leq \infty$, the weighted Fock–Sobolev spaces $\mathcal{F}^p_{(m,\alpha)}$ of order m consists of entire functions f on $\mathbb{C}^n$ such that

$$\|f\|_{(p,m)} = \sum_{\beta_{sn}\leq m} \|\partial^\beta f\|_p < \infty \tag{1.1}$$

where $\beta_{sn} = \beta_1 + \beta_2 + ... + \beta_n$. These spaces have recently been introduced by R. Cho and K. Zhu [9], and one of their main results provides the following useful Fourier characterization of the spaces.

Lemma 1.1. *Let $0 < p \leq \infty$. Then an entire function f on $\mathbb{C}^n$ belongs to $\mathcal{F}^p_{(m,\alpha)}$ if and only if $z^\beta f$ belongs to $\mathcal{F}^p_\alpha$ for all nonnegative multi-indices β with $\beta_{sn} = m$ where $z^\beta = z_1^{\beta_1} z_2^{\beta_2} z_3^{\beta_3} ... z_n^{\beta_n}$.*

As a consequence of this lemma, it was proved that the norm in (1.1) is comparable to

$$\|f\|_{(p,m)} = \Big(C_{(p,m,n)} \int_{\mathbb{C}^n} |z|^{mp}|f(z)|^p e^{-\frac{\alpha p}{2}|z|^2} dV(z)\Big)^{1/p} \tag{1.2}$$

for $0 < p < \infty$ and

$$C_{(p,m,n)} = \Big(\frac{\alpha p}{2}\Big)^{(mp/2)+n} \frac{\Gamma(n)}{\pi^n \Gamma((mp)/2 + n)},$$

where Γ denotes the Gamma function. For $p = \infty$, the corresponding norm is

$$\|f\|_{(\infty,m)} = \sup_{z\in\mathbb{C}^n} |z|^m |f(z)| e^{-\frac{\alpha}{2}|z|^2}. \tag{1.3}$$

We find it more convenient to use this equivalent norm through out the rest of the paper. We note in passing that the Fock–Sobolev spaces of order m can also be considered as a weighted (generalized) Fock spaces $\mathcal{F}^p_{\varphi_m}$ consisting of entire functions f for which

$$\Big(\frac{\alpha p}{2\pi}\Big)^n \int_{\mathbb{C}^n} |f(z)|^p e^{-p\varphi_m(z)} dV(z) < \infty$$

for $0 < p < \infty$ and $\sup_{z\in\mathbb{C}^n} |f(z)| e^{-\varphi_m(z)} < \infty$ for $p = \infty$ where $\varphi_m(z) = -m\log(1 + |z|) + \alpha|z|^2/2$.

We next recall the notion of lattice for the space $\mathbb{C}^n$. For a positive r we denote by $D(z, r)$ the set $\{w \in \mathbb{C}^n : |z - w| < r\}$. We say that a sequence of distinct points $(z_k)_{k\in\mathbb{N}} \subset \mathbb{C}^n$ is an $r/2-$ lattice for $\mathbb{C}^n$ if the sequence of the balls $D(z_k, r)$, $k \in \mathbb{N}$ constitutes a covering of $\mathbb{C}^n$ and the balls $D(z_k, r/2)$ are mutually disjoint. The sequence $(z_k), k \in \mathbb{N}$ will refer to such $r/2$ lattice

with a fixed r in the remaining part of the paper. An interesting example of such a lattice can be found in [15].

Lemma 1.2. *Let $r > 0$ and $(z_k)_{k\in\mathbb{N}}$ be an $r/2-$ lattice for $\mathbb{C}^n$. Then there exists a positive integer $N_{\max}$ such that every point in $\mathbb{C}^n$ belongs to at most $N_{\max}$ of the balls $D(z_k, 2r)$.*

The proof of the lemma can be found in [24, 33] where in [24] a more general setting has been considered.

Let μ be a positive Borel measure on $\mathbb{C}^n$. Then its average on $D(z, r)$ is the quantity $\mu(D(z, r))/Vol(D(z, r))$ where $Vol(D(z, r))$ is the Euclidean volume of the ball which is a constant for all z in $\mathbb{C}^n$. In what follows, we simply refer $\mu(D(., r))$ as an averaging function of μ, and $\mu(D(z_k, r))$ as its averaging sequence.

A word on notation: The notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant C such that $U(z) \leq CV(z)$ holds for all z in the set in question, which may be a Hilbert space or a set of complex numbers. We write $U(z) \simeq V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$.

2. The (p, q) Fock–Carleson measures on Fock–Sobolev spaces

Carleson measures were first introduced by L. Carleson [4] as a tool to study interpolating sequences in the Hardy space H^∞ of bounded analytic functions in the unit disc and the corona problem. Since then the measures have found numerous applications and extensions in the study of various spaces of functions: for example see [1, 2, 3, 7, 10, 11, 18, 23, 27]. In this paper, we study one of its extensions namely the (p, q) Fock–Carleson measures for weighted Fock–Sobolev spaces. In the next section, we will also look at application of such measures in studying some mapping properties of weighted composition operators acting between different weighted Fock–Sobolev spaces.

Let $0 < p \leq \infty$ and $0 < q < \infty$. Then we call a nonnegative measure μ on $\mathbb{C}^n$ a (p, q) Fock–Carleson measure for Fock–Sobolev spaces if[1]

$$\int_{\mathbb{C}^n} |f(z)|^q e^{\frac{-\alpha q}{2}|z|^2} d\mu(z) \lesssim \|f\|^q_{(p,m)} \tag{2.1}$$

for all f in $\mathcal{F}^p_{(m,\alpha)}$. In other words, μ is a (p, q) Fock–Carleson measure if and only if the canonical embedding map $I_\mu : \mathcal{F}^p_{(m,\alpha)} \to L^q(\sigma_q)$ is bounded where $d\sigma_q(z) = e^{-\frac{q\alpha}{2}|z|^2} d\mu(z)$. We call μ a (p, q) vanishing Fock–Carleson measure if

$$\lim_{j\to\infty} \int_{\mathbb{C}^n} |f_j(z)|^q e^{\frac{-q\alpha}{2}|z|^2} d\mu(z) = 0$$

whenever f_j is a uniformly bounded sequence in $\mathcal{F}^p_{(m,\alpha)}$ that converges uniformly to zero on compact subsets of $\mathbb{C}^n$ as $j \to \infty$. We will write $\|\mu\| = \|I_\mu\|$ for the smallest admissible constant in inequality (2.1) which often is called the Carleson constant.

[1]We follow the approach to Carleson measures taken in [9].

For $s, t > 0$, we may define the (t, s)-Berezin type transform of μ by

$$\widetilde{\mu}_{(t,s)}(w) = \int_{\mathbb{C}^n} (1 + |z|)^{-s} e^{-\frac{t\alpha}{2}|z-w|^2} d\mu(z).$$

As will be seen, its role is analogous to that played by the Berezin transform for the Bergman spaces. For convenience, we will also use the notations

$$\mu_s(z) = \frac{\mu(z)}{(1 + |z|)^s}, \; \mu_{(s,r,D)}(z) = \frac{\mu(D(z, r))}{(1 + |z|)^s}, \text{ and } \; L^p = L^p(\mathbb{C}^n, dV).$$

We may now state our first main result.

Theorem 2.1. *Let* $0 < p \le q < \infty$ *and* $\mu \ge 0$. *Then the following statements are equivalent.*

(i) μ *is a* (p, q) *Fock–Carleson measure;*
(ii) $\widetilde{\mu}_{(t,mq)} \in L^\infty$ *for some (or any)* $t > 0$;
(iii) $\mu_{(mq,r,D)} \in L^\infty$ *for some (or any)* $r > 0$;
(iv) $\mu_{(mq,r,D)}(z_k) \in \ell^\infty$ *for some (or any)* $r > 0$. *Moreover, we have*

$$\|\mu\|^q \simeq \|\widetilde{\mu}_{(t,mq)}\|_{L^\infty} \simeq \|\mu_{(mq,r,D)}\|_{L^\infty} \simeq \|\mu_{(mq,r,D)}(z_k)\|_{\ell^\infty}. \tag{2.2}$$

Vanishing Carleson measures appear naturally in the study of compact composition operators, Toeplitz and Hankel operators, Volterra type integral operators, two weight Hilbert transforms, and in several other contexts in various functional spaces. As far as their characterization is concerned, there exists a general "folk theorem": once the Carleson measures are described by a certain "big oh" condition, vanishing Carleson measures are then characterized by the corresponding "little oh" counterparts. This does not however mean that such " folk theorem" is always true. See [6] for a counterexample. Our next result shows that it still holds on Fock–Sobolev spaces.

Theorem 2.2. *Let* $0 < p \le q < \infty$ *and* $\mu \ge 0$. *Then the following statements are equivalent.*

(i) μ *is a* (p, q) *vanishing Fock–Carleson measure;*
(ii) $\widetilde{\mu}_{(t,mq)}(z) \to 0$ *as* $|z| \to \infty$ *for some (or any)* $t > 0$;
(iii) $\mu_{(mq,r,D)}(z) \to 0$ *as* $|z| \to \infty$ *for some (or any)* $r > 0$;
(iv) $\mu_{(mq,r,D)}(z_k) \to 0$ *as* $k \to \infty$ *for some (or any)* $r > 0$.

Conditions (ii), (iii) and (iv) in the two theorems above are independent of the parameter α and exponent $p \le q$. It means that if μ is a (p, q) (vanishing) Fock–Carleson measure for some $p \le q$ and $\alpha > 0$, then it is a (p_1, q) (vanishing) Fock–Carleson measure for any $p_1 \le q$ and every other parameter α. On the other hand, the conditions are dependent on the size of the exponent q in the target space in the sense that if μ constitutes a (p, q) Fock–Carleson measure for some $q \ge p$, then it may fail to be a (p, q_1) Fock measures for any $q_1 \ge p$ unless $m = 0$ or $q_1 \ge q$. This presents a clear distinction with the corresponding conditions for the ordinary Fock spaces ($m = 0$). Because, in the later, it holds that μ is a (p, q) Fock–Carleson measure for some $p \le q$ if and only if it is a (p_1, q_1) Fock–Carleson measure for any pair of exponents (p_1, q_1) for which $p_1 \le q_1$. If we take a different

approach to the (p, q) measures and redefine inequality (2.1) by replacing $d\mu(z)$ with $(1+|z|)^{mq}d\mu(z)$ the distinction mentioned above would disappear and the (p,q) measure conditions will be exactly the same as they are for ordinary Fock spaces.

As in the case of ordinary Fock spaces, the Fock–Sobolev spaces satisfy the inclusion monotonicity property $\mathcal{F}^p_{(m,\alpha)} \subseteq \mathcal{F}^q_{(m,\alpha)}$ whenever $0 < p \leq q \leq \infty$ [9]. Thus, for $p > q$, the boundedness conditions on the averaging functions, averaging sequences and (t, mq)-Berezin transforms will be replaced by the next stronger $p/(p-q)$ integrability against a weight of polynomial growth conditions.

Theorem 2.3. *Let $0 < q < p < \infty$ and $\mu \geq 0$. Then the following statements are equivalent.*

(i) *μ is a (p, q) Fock–Carleson measure;*
(ii) *μ is a (p, q) vanishing Fock–Carleson measure;*
(iii) *$\widetilde{\mu}_{(t,mq)} \in L^{\frac{p}{p-q}}$ for some (any) $t > 0$;*
(iv) *$\mu_{(mq,r,D)} \in L^{\frac{p}{p-q}}$ for some (or any) $r > 0$;*
(v) *$\mu_{(mq,r,D)}(z_k) \in \ell^{\frac{p}{p-q}}$ for some (or any) $r > 0$. Moreover, we have*

$$\|\mu\|^q \simeq \|\mu_{(mq,r,D)}\|_{L^{\frac{p}{p-q}}} \simeq \|\widetilde{\mu}_{(t,mq)}\|_{L^{\frac{p}{p-q}}} \simeq \|\mu_{(mq,r,D)}(z_k)\|_{\ell^{\frac{p}{p-q}}}. \tag{2.3}$$

Observe that the fraction $p/(p-q)$ is the conjugate exponent of p/q whenever $0 < q \leq p < \infty$. In the limiting case, i.e., when $p = \infty$, the next yet stronger condition holds.

Theorem 2.4. *Let $0 < q < \infty$ and $\mu \geq 0$. Then the following statements are equivalent.*

(i) *μ is an (∞, q) Fock–Carleson measure;*
(ii) *μ is an (∞, q) vanishing Fock–Carleson measure;*
(iii) *$\widetilde{\mu}_{(t,mq)} \in L^1$ for some(or any) $t > 0$;*
(iv) *$\mu_{(mq,r,D)} \in L^1$ for some (or any) $r > 0$;*
(v) *$\mu_{(mq,r,D)}(z_k) \in \ell^1$ for some (or any) $r > 0$;*
(vi) *μ_{mq} is a finite measure on $\mathbb{C}^n$. Moreover, we have*

$$\|\mu\|^q \simeq \|\widetilde{\mu}_{(t,mq)}\|_{L^1} \simeq \|\mu_{(mq,r,D)}\|_{L^1} \simeq \mu_{mq}(\mathbb{C}^n) \simeq \|\mu_{(mq,r,D)}(z_k)\|_{\ell^1}. \tag{2.4}$$

The four results above unify and extend a number of recent results in the area. For example when $m = 0$, while the first three of the results simplify to results obtained in [15], Theorem 2.4 simplifies to a result obtained in [22]. On the other hand, when $p = q$ the first two theorems give Theorem 21 and Theorem 22 of [9]. If $m = 0$ and $p = q = 2$, then the first two theorems again simplify to results obtained in [16].

3. Weighted composition operators on Fock–Sobolev spaces

Let $H(\mathbb{C}^n)$ denotes the space of entire functions on $\mathbb{C}^n$. Each pair of entire functions (ψ, u) induces a weighted composition operator $uC_\psi f = u(f \circ \psi)$

on $H(\mathbb{C}^n)$. Questions about boundedness, compactness, and other operator theoretic properties of uC_ψ expressed in terms of function theoretic conditions on u and ψ have been a subject of high interest, and have been studied by several authors in various function spaces. The Schatten class membership properties of uC_ψ on $\mathcal{F}^2_{(m,\alpha)}$ has recently been studied in [21]. In this section, we will study the bounded and compact mapping properties of uC_ψ when it acts between different weighted Fock–Sobolev spaces. We will also estimate the norm and essential norm of uC_ψ in terms of certain Berezin type integral transforms. The approach we intend to follow links some of these properties of uC_ψ with the (p,q) Fock–Carleson measures which allows us to apply the results obtained in the previous section. Indeed, this offers a simple example where the (p,q) Fock–Carleson measures find some applications in operator theory.

Our results on uC_ψ will be expressed in terms of the function

$$B^\infty_{(m,\psi)}(|u|)(z) = \frac{|z|^m|u(z)|}{(1+|\psi(z)|)^m} e^{\frac{\alpha}{2}\left(|\psi(z)|^2-|z|^2\right)}$$

and a Berezin type integral transform

$$B_{(m,\psi)}(|u|^p)(w) = \int_{\mathbb{C}^n} \frac{|k_w(\psi(z))|^p}{(1+|\psi(z)|)^{mp}} |u(z)|^p|z|^{pm} e^{-\frac{\alpha p}{2}|z|^2} dV(z).$$

3.1. Bounded and compact uC_ψ

Theorem 3.1. *Let $0 < p \le q < \infty$ and (u,ψ) be a pair of entire functions. Then $uC_\psi : \mathcal{F}^p_{(m,\alpha)} \to \mathcal{F}^q_{(m,\alpha)}$ is*

(i) *bounded if and only if $B_{(m,\psi)}(|u|^q)$ belongs to L^∞. Moreover, we have*

$$\|uC_\psi\| \simeq \|B_{(m,\psi)}(|u|^q)\|^{1/q}_{L^\infty}. \tag{3.1}$$

(ii) *compact if and only if*

$$\lim_{|z|\to\infty} B_{(m,\psi)}(|u|^q)(z) = 0.$$

Note that like in Theorem 2.1, the conditions both in (i) and (ii) are independent of the exponent p apart from the fact that p should not be exceeding q. In other words, if there exists a $p > 0$ for which uC_ψ is bounded (compact) from $\mathcal{F}^p_{(m,\alpha)}$ to $\mathcal{F}^q_{(m,\alpha)}$, then it is also bounded (compact) from $\mathcal{F}^{p_1}_{(m,\alpha)}$ to $\mathcal{F}^q_{(m,\alpha)}$ for every other $p_1 \le q$.

A natural question is whether there exists an interplay between the two symbols u and ψ in inducing bounded and compact operators uC_ψ. We first observe that by the classical Liouville's theorem a nonconstant function u can not decay. The following is a simple consequence of this fact.

Corollary 3.2. *Let $0 < p \le q < \infty$ and (u,ψ) be a pair of entire functions. If $u \ne 0$ and $uC_\psi : \mathcal{F}^p_{(m,\alpha)} \to \mathcal{F}^q_{(m,\alpha)}$ is bounded, then $\psi(z) = Az + B$ where A is an $n\times n$ matrix, $\|A\| \le 1$ and B is an $n \times 1$ matrix such that $\langle Aw, B\rangle = 0$ whenever $|Aw| = |w|$ for some $w \in \mathbb{C}^n$. Moreover, if uC_ψ is compact, then $\|A\| < 1$ where $\|A\|$ refers to the operator norm of matrix A.*

By setting $u = 1$ and simplifying the conditions in Theorem 3.1, one can easily see that the linear forms for ψ are both necessary and sufficient for $C_\psi : \mathcal{F}^p_{(m,\alpha)} \to \mathcal{F}^q_{(m,\alpha)}$ to be bounded (compact). This fact together with Corollary 3.2 ensures that boundedness of uC_ψ implies boundedness of C_ψ while the converse in general fails. The same conclusion holds for compactness. Particular cases of theses conclusions could be also read in [5, 8].

Proof. Observe that

$$B_{(m,\psi)}(|u|^q)(z) \geq \int_{\mathbb{C}^n} \frac{|k_w(\psi(z))|^q}{(1+|\psi(z)|)^{mq}} |u(z)|^q |z|^{qm} e^{-\frac{\alpha q}{2}|z|^2} dV(z)$$
$$\gtrsim \frac{|k_w(\psi(z))|^q}{(1+|\psi(z|)^{mq}} |u(w)|^q |z|^{qm} e^{-\frac{\alpha q}{2}|z|^2}$$

for all $z, w \in \mathbb{C}^n$. Applying Theorem 3.1 and setting $w = \psi(z)$ in particular gives

$$\infty > \sup_{w\in\mathbb{C}^n} \frac{|k_w(\psi(z))|^q |u(z)|^q |z|^{qm}}{(1+|\psi(z|)^{mq} e^{\frac{\alpha q}{2}|w|^2}} \geq \frac{|u(z)|^q |z|^{qm}}{(1+|\psi(z)|)^{mq}} e^{\frac{\alpha q}{2}|\psi(z)|^2 - |z|^2}. \tag{3.2}$$

Indeed, we claim that

$$\limsup_{|z|\to\infty} \frac{|\psi(z)|}{|z|} \leq 1. \tag{3.3}$$

We argue by contradiction, and suppose (3.3) fails. Then there exists a sequence (z_j) such that $|z_j| \to \infty$ as $j \to \infty$ and

$$\limsup_{|z_j|\to\infty} |\psi(z_j)|/|z_j| > 1.$$

For nobility, we set $w_j = |\psi(z_j)|/|z_j|$, and observe

$$\limsup_{j\to\infty} M_\infty(u^q, (1+|\psi(z_j)|)^{mq}) \lesssim \limsup_{j\to\infty} \frac{1}{|z_j|^{qm} e^{\frac{\alpha q}{2}(|\psi(z_j)|^2 - |z_j|^2)}}$$
$$= \limsup_{j\to\infty} \frac{1}{|z_j|^{mq} e^{\frac{\alpha q}{2}|z_j|^2(w_j^2 - 1)}} = 0 \tag{3.4}$$

which gives a contradiction as u is a constant entire function and $M_\infty(u^q, (1+|\psi(z_j)|)^{mq})$ is the integral mean of the function $|u|^q$. Thus, (3.3) implies $\psi(z) = Az + B$ for some A an $n \times n$ matrix with $\|A\| < 1$ and $B \in \mathbb{C}^n$.

Let now η be a point in $\mathbb{C}^n$ such that $|A\eta| = |\eta|$. We may further assume that $|\eta| = 1$ and $A\eta = \eta$ where the latter is due to unitary change of variables; see the proof of [5, Theorem 1]. If $z = t\tau\eta$ where $|\tau| = 1$ is a constant for which $\tau\langle A\eta, B\rangle = |\langle A\eta, B\rangle|$, then

$$\frac{|u(z)|^q |z|^{qm}}{(1+|\psi(z)|)^{mq}} e^{\frac{\alpha q}{2}|\psi(z)|^2 - |z|^2} = \frac{|u(t\tau\eta)| e^{\frac{\alpha q}{2}(|B|^2 + 2t|\langle A\eta, B\rangle|)}}{1 + t^{-2}|B|^2 + 2t^{-1}|\langle A\eta, B\rangle|}. \tag{3.5}$$

By (3.2), the fraction (3.5) has to be finite as $t \to \infty$, and this holds only if $\langle A\eta, B\rangle = 0$ as desired.

If, in addition, uC_ψ is compact, then by part (ii) of Theorem 3.1,

$$\lim_{|z|\to\infty} \frac{|u(z)|^q |z|^{qm}}{(1+|\psi(z)|)^{mq}} e^{\frac{\alpha q}{2}|\psi(z)|-|z|^2} = 0. \tag{3.6}$$

A simple modification of the above arguments show that (3.6) holds only if $\psi(z) = Az + B$ with $\|A\| < 1$. $\square$

We now consider the case where $p > q$. Mapping $\mathcal{F}^p_{(m,\alpha)}$ into $\mathcal{F}^q_{(m,\alpha)}$ gives the following stronger integrability conditions as one would expect.

Theorem 3.3. *Let $0 < q < p < \infty$ and (u,ψ) be a pair of entire functions. Then the following statements are equivalent.*

(i) $uC_\psi : \mathcal{F}^p_{(m,\alpha)} \to \mathcal{F}^q_{(m,\alpha)}$ *is bounded;*
(ii) $uC_\psi : \mathcal{F}^p_{(m,\alpha)} \to \mathcal{F}^q_{(m,\alpha)}$ *is compact;*
(iii) $B_{(m,\psi)}(|u|^q)$ *belongs to* $L^{\frac{p}{p-q}}$. *We further have the norm estimate*

$$\|uC_\psi\| \simeq \|B_{(m,\psi)}(|u|^q)\|^{\frac{p-q}{p}}_{L^{\frac{p}{p-q}}}. \tag{3.7}$$

Following a similar approach as in the proof of Corollary 3.2, we observe that the $L^{p/(p-q)}$ integrability of $B_{(m,\psi)}(|u|^q)$ restricts further ψ to have only the linear form $\psi(z) = Az + B$ with $\|A\| < 1$.

Theorem 3.4. *Let $0 < q < \infty$ and (u,ψ) be a pair of entire functions. Then the following statements are equivalent.*

(i) $uC_\psi : \mathcal{F}^\infty_{(m,\alpha)} \to \mathcal{F}^q_{(m,\alpha)}$ *is bounded;*
(ii) $uC_\psi : \mathcal{F}^\infty_{(m,\alpha)} \to \mathcal{F}^q_{(m,\alpha)}$ *is compact;*
(iii) $B_{(m,\psi)}(|u|^q)$ *belongs to* L^1. *Furthermore, we have the asymptotic norm estimate*

$$\|uC_\psi\| \simeq \|B_{(m,\psi)}(|u|^q)\|^{1/q}_{L^1}. \tag{3.8}$$

As it will be seen in the proofs, the boundedness and compactness conditions for uC_ψ in Theorems 3.1-3.4 can be equivalently expressed in terms of (p,q) (vanishing) Fock–Carleson measures, averaging functions, and averaging sequences of appropriately chosen positive measures μ on $\mathbb{C}^n$. .

Theorem 3.5. *Let $0 < p \le \infty$ and (u,ψ) be a pair of entire functions. Then $uC_\psi : \mathcal{F}^p_{(m,\alpha)} \to \mathcal{F}^\infty_{(m,\alpha)}$ is*

(i) *bounded if and only if $B^\infty_{(m,\psi)}(|u|)$ belongs to L^∞. Moreover, we have*

$$\|uC_\psi\| \simeq \|B^\infty_{(m,\psi)}(|u|)\|_{L^\infty}. \tag{3.9}$$

(ii) *compact if and only if it is bounded and*

$$\lim_{|\psi(z)|\to\infty} B^\infty_{(m,\psi)}(|u|)(z) = 0. \tag{3.10}$$

An interesting observation is to replace $\mathcal{F}^\infty_{(m,\alpha)}$ by a smaller space $\mathcal{F}^\infty_{(0,m,\alpha)}$; consisting of all analytic function f such that

$$\lim_{|z|\to\infty} |f(z)||z|^m e^{-\frac{\alpha}{2}|z|^2} = 0.$$

The space $\mathcal{F}^\infty_{(0,m,\alpha)}$ constitutes a proper Banach subspace of $\mathcal{F}^\infty_{(m,\alpha)}$ and contains the spaces $\mathcal{F}^p_{(m,\alpha)}$ for all $p<\infty$. If we replace $\mathcal{F}^p_{(m,\alpha)}$ with a larger space $\mathcal{F}^\infty_{(0,m,\alpha)}$ in part (i) of the preceding theorem, the condition remains unchanged. On the other hand, modifying the arguments used to prove part (ii) of the theorem shows that the following stronger condition holds when we replace the target space with $\mathcal{F}^\infty_{(0,m,\alpha)}$.

Corollary 3.6. *Let $0<p<\infty$ and (u,ψ) be a pair of entire functions. Then the map $uC_\psi : \mathcal{F}^p_{(m,\alpha)}($ or $\mathcal{F}^\infty_{(0,m,\alpha)}) \to \mathcal{F}^\infty_{(0,m,\alpha)}$ is compact if and only if*

$$\lim_{|z|\to\infty} B^\infty_{(m,\psi)}(|u|)(z)=0. \tag{3.11}$$

We may mention that for the special case $m=0$, Theorem 3.5 and its corollary were proved in [26].

3.2. Essential norm of uC_ψ

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Banach spaces. Then the essential norm $\|T\|_e$ of a bounded operator $T:\mathscr{H}_1\to\mathscr{H}_2$ is defined as the distance from T to the space of compact operators from $\mathscr{H}_1$ and $\mathscr{H}_2$. We refer readers to [12, 13, 20, 25, 26, 29, 30, 31] for estimations of such norms for different operators on Hardy space, Bergman space, L^p and some Fock spaces. We get the following estimate for uC_ψ when it acts on weighted Fock–Sobolev spaces.

Theorem 3.7. *Let $1<p\le q\le\infty$ and $p\neq\infty$. If $uC_\psi:\mathcal{F}^p_\alpha\to\mathcal{F}^q_\alpha$ is bounded, then*

$$\|uC_\psi\|_e \simeq \begin{cases} \left(\limsup_{|w|\to\infty} B_{(\psi,m)}(|u|^q)(w)\right)^{\frac{1}{q}}, & q<\infty \\ \limsup_{|\psi(w)|\to\infty} B^\infty_{(\psi,m)}(|u|)(w), & q=\infty. \end{cases} \tag{3.12}$$

For $p>1$, the compactness conditions in Theorem 3.1 and Theorem 3.5 could be easily drawn from this relation since the left-hand side expression in (3.12) in this case vanishes for compact uC_ψ.

All the results obtained on uC_ψ again unify and generalize a number of recent results in the area including from [5, 8, 20, 22, 26, 29, 30]. One may simply set $m=0$ and simplify the conditions to get the classical known results on the ordinary Fock spaces.

We also mention that we have not explicitly used the kernel function $K_{(w,m)}$ for the space $\mathcal{F}^2_{(m,\alpha)}$, $m\neq 0$ in dealing with any of the results presented here. Finding an explicit expression for $K_{(w,m)}$ is still an open problem. By Corollary 13 of [9] we have in moduli that

$$|K_{(w,m)}(z)| \lesssim \frac{\|K_z\|_2\|K_w\|_2}{e^{\frac{\alpha}{8}|z-w|^2}(1+|z||w|)^m}.$$

It remains open whether the reverse estimate above holds. On the other hand, it was proved in [8] that

$$\|K_{(w,m)}\|^2_{(m,2)} \simeq \frac{\|K_w\|_2^2}{(1+|w|)^{2m}}.$$

4. Some auxiliary lemmas

In this section we prove some lemmas which play key roles in our next considerations. The lemmas are also interest of their own. For a given measurable function f and a Borel measure μ_f on $\mathbb{C}^n$ such that $d\mu_f(z) = f(z)dV(z)$, we prove the following.

Lemma 4.1. *let $1 \leq p \leq \infty$ and $0 < r, t, s < \infty$. Then the maps $f \mapsto f_{(r,s,D)}$ and $f \mapsto \widetilde{f}_{(t,s)}$ are bounded on L^p where $f_{(r,s,D)}(z) := (1+|z|)^{-s}\mu_f(D(z,r))$ and*

$$\widetilde{f}_{(t,s)}(z) := \int_{\mathbb{C}^n} (1+|w|)^{-s} e^{-\frac{\alpha t}{2}|w-z|^2} d\mu_f(w).$$

Proof. We mention that for the case when $s = 0$, the lemma was first proved in [15]. Using the additional fact that

$$\sup_{w \in \mathbb{C}^n} (1+|w|)^{-s} \leq 1, \tag{4.1}$$

for all nonnegative s and t, the arguments there can be easily adopted to work for all positive s. We outline the proof as follows. We use interpolation argument on L^p Lebesgue spaces. Thus, it suffices to establish the statements for $p = 1$ and $p = \infty$. We begin with the case $p = 1$. Using (4.1) and Fubini's theorem, we have

$$\begin{aligned} \|\widetilde{f}_{(t,s)}\|_{L^1} &= \int_{\mathbb{C}^n} \left| \int_{\mathbb{C}^n} (1+|w|)^{-s} e^{-\frac{\alpha t}{2}|w-z|^2} d\mu_f(w) \right| dV(z) \\ &= \int_{\mathbb{C}^n} \left(\int_{\mathbb{C}^n} e^{-\frac{\alpha t}{2}|w-z|^2} dV(z) \right) \frac{|f(w)|}{(1+|w|)^s} dV(w) \\ &\simeq \int_{\mathbb{C}^n} \frac{|f(w)|}{(1+|w|)^s} dV(w) \lesssim \|f\|_{L^1}. \end{aligned}$$

Applying again (4.1) for $t = 1$, Fubini's theorem, and the fact that $\chi_{D(\zeta,r)}(z) = \chi_{D(z,r)}(\zeta)$ for all ζ and z in $\mathbb{C}^n$, we have

$$\begin{aligned} \|f_{(r,s,D)}\|_{L^1} = \int_{\mathbb{C}^n} (1+|z|)^{-s} \mu_f(D(z,r)) dV(z) &\leq \int_{\mathbb{C}^n} \int_{D(z,r)} |f(\zeta)| dV(\zeta) dV(z) \\ &= \int_{\mathbb{C}^n} |f(\zeta)| \int_{\mathbb{C}^n} \chi_{D(\zeta,r)}(z) dV(z) dV(\zeta) \simeq \|f\|_{L^1}. \end{aligned}$$

On the other hand, for $p = \infty$ it easily follows that

$$\begin{aligned} \|f_{(r,s,D)}\|_{L^\infty} = \sup_{z \in \mathbb{C}^n} \left| (1+|z|)^{-s} \mu_f(D(z,r)) \right| &\leq \sup_{z \in \mathbb{C}^n} \int_{D(z,r)} |f(\zeta)| dV(\zeta) \\ &\leq \|f\|_{L^\infty} \sup_{z \in \mathbb{C}^n} \int_{D(z,r)} dV(\zeta) \lesssim \|f\|_{L^\infty}. \end{aligned}$$

Seemingly, for each $f \in L^\infty$, we also have

$$\begin{aligned}\|\widetilde{f}_{(t,s)}\|_{L^\infty} &= \sup_{z\in\mathbb{C}^n}\int_{\mathbb{C}^n} e^{-\frac{t\alpha}{2}|w-z|^2}(1+|w|)^{-s}|f(w)|dV(\zeta)\\ &\leq \sup_{z\in\mathbb{C}^n}\int_{\mathbb{C}^n} e^{-\frac{t\alpha}{2}|z-w|^2}|f(\zeta)|dV(w)\\ &\leq \|f\|_{L^\infty}\sup_{z\in\mathbb{C}^n}\int_{\mathbb{C}^n} e^{-\frac{t\alpha}{2}|z-w|^2}dV(w) \lesssim \|f\|_{L^\infty}, \end{aligned} \tag{4.2}$$

and completes the proof. □

Lemma 4.2. *Let* $1 \leq p \leq \infty$, $0 < s < \infty$ *and* $\mu \geq 0$ *be a measure on* $\mathbb{C}^n$. *Then if* $\mu_{(s,\delta,D)}$ *belongs to* L^p *for some* $\delta > 0$, *then so does* $\mu_{(s,r,D)}$ *for all* $r > 0$.

Proof. For each τ in $\mathbb{C}^n$ we may write

$$\int_{D(\tau,r)}(1+|z|)^{-s}\mu(D(z,\delta))dV(z) = \int_{D(\tau,r)}\int_{\mathbb{C}^n}(1+|z|)^{-s}\chi_{D(z,\delta)}(\zeta)d\mu(\zeta)dV(z).$$

Using again the simple fact that $\chi_{D(z,\delta)}(\zeta) = \chi_{D(\zeta,\delta)}(z)$, the double integral above is easily seen to be equal to

$$\begin{aligned}\int_{\mathbb{C}^n}\int_{D(\zeta,\delta)\cap D(\tau,r)}\frac{dV(z)d\mu(\zeta)}{(1+|z|)^s} &\simeq \int_{\mathbb{C}^n}\frac{Vol\big(D(\zeta,\delta)\cap D(\tau,r)\big)}{(1+|\tau|)^s}d\mu(\zeta)\\ &\geq \frac{1}{(1+|\tau|)^s}\int_{D(\tau,r)} Vol\big(D(\zeta,\delta)\cap D(\tau,r)\big)d\mu(\zeta)\end{aligned}$$

where $Vol(E)$ refers to the Lebesque measure of set $E \subset \mathbb{C}^n$. Clearly, the right hand quantity is bounded from below by

$$(1+|\tau|)^{-s}\mu(D(\tau,r))\inf_{\zeta\in D(\tau,r)} Vol\big(D(\zeta,\delta)\cap D(\tau,r)\big) \gtrsim (1+|\tau|)^{-s}\mu(D(\tau,r))$$

where the lower estimate follows here since $\zeta \in D(\tau,r)$, there obviously exists a ball $D(\tau_0,r_0)$ contained in $D(\zeta,\delta)\cap D(\tau,r)$ with $Vol\big(D(\tau_0,r_0)\big) \simeq r_0^n$. From the above analysis, we conclude

$$(1+|\tau|)^{-s}\mu(D(\tau,r)) \lesssim \int_{D(\tau,r)}(1+|z|)^{-s}\mu(D(z,\delta))dV(z). \tag{4.3}$$

If we now set $f(z) = (1+|z|)^{-s}\mu(D(z,\delta))$, then the estimate above along with Lemma 4.1 ensure that

$$\|\mu_{(s,r,D)}\|_{L^p} \lesssim \|f_{(s,\delta)}\|_{L^p} \lesssim \|f\|_{L^p} = \|\mu_{(s,\delta,D)}\|_{L^p} < \infty \tag{4.4}$$

for each $p \geq 1$ and any $r > 0$. □

Our next lemma gives the link among averaging sequence, averaging functions and Berezin type integral transform of a given measure.

Lemma 4.3. *Let* $1 \leq p \leq \infty$ *and* $0 < s < \infty$. *Then*

$$\|\mu_{(s,r,D)}\|_{L^p} \simeq \|\widetilde{\mu}_{(t,s)}\|_{L^p} \simeq \|\mu_{(s,r,D)}(z_k)\|_{\ell^p} \tag{4.5}$$

for some (or any) $r > 0$ *and* $t > 0$.

Proof. We begin by noting that since $\widetilde{\mu}_{(t,s)}$ is independent of r, if the estimate in (4.5) holds for some $r > 0$, then it holds for every other positive r. The same holds with t as $\mu_{(s,r,D)}$ is independent of it. The proof of the lemma follows from a careful modification of some arguments used in the proof of Theorem 13 in [16]. We may first observe that for each w in the ball $D(z,r)$, the estimate

$$1 + |z| \simeq 1 + |w| \tag{4.6}$$

holds. We proceed to show the first estimate in (4.5). Using (4.6) we have

$$\begin{aligned}\frac{\mu(D(z,r))}{(1+|z|)^s} = \frac{1}{(1+|z|)^s}\int_{D(z,r)} d\mu(w) &\leq \frac{e^{\frac{\alpha t r^2}{2}}}{(1+|z|)^s}\int_{D(z,r)} e^{-\frac{\alpha t}{2}|z-w|^2} d\mu(w)\\ &\lesssim e^{\frac{\alpha t r^2}{2}}\int_{D(z,r)} \frac{e^{-\frac{\alpha t}{2}|z-w|^2}}{(1+|w|)^s} d\mu(w) \lesssim \widetilde{\mu}_{(t,s))}(z)\end{aligned}$$

from which we get

$$\|\mu_{(s,r,D)}\|_{L^p} \lesssim \|\widetilde{\mu}_{(t,s)}\|_{L^p} \tag{4.7}$$

for each $p \geq 1$. On the other hand, by Lemma 1 of [16], we note that the pointwise estimate

$$|f(z)e^{-\frac{\alpha}{2}|z|^2}|^q \lesssim \int_{D(z,r)} |f(w)|^q e^{-\frac{\alpha q}{2}|w|^2} dV(w) \tag{4.8}$$

holds for any f in $H(\mathbb{C}^n)$, $q, r > 0$ and z in $\mathbb{C}^n$. From this and (4.6), we deduce

$$\begin{aligned}\frac{|f(z)e^{-\frac{\alpha}{2}|z|^2}|^q}{(1+|z|)^s} &\lesssim \frac{1}{(1+|z|)^s}\int_{D(z,r)} |f(w)e^{-\frac{\alpha}{2}|w|^2}|^q dV(w)\\ &\simeq \int_{D(z,r)} \frac{|f(w)e^{-\frac{\alpha}{2}|w|^2}|^q}{|(1+|w|)^s} dV(w)\end{aligned}$$

Integrating the above against the measure μ, we find

$$\begin{aligned}\int_{\mathbb{C}^n} \frac{|f(z)e^{-\frac{\alpha}{2}|z|^2}|^q}{(1+|z|)^s} d\mu(z) &\lesssim \int_{\mathbb{C}^n}\int_{D(z,r)} \frac{|f(w)e^{-\frac{\alpha}{2}|w|^2}|^q}{(1+|w|)^s} dV(w) d\mu(z)\\ &= \int_{\mathbb{C}^n} \frac{|f(w)e^{-\frac{\alpha}{2}|w|^2}|^q}{(1+|w|)^s}\int_{\mathbb{C}^n} \chi_{D(w,r)}(z) dV(w) d\mu(z).\end{aligned}$$

It follows from this estimate and Fubini's theorem that

$$\int_{\mathbb{C}^n} \frac{|f(w)e^{-\frac{\alpha}{2}|w|^2}|^q}{(1+|w|)^s} d\mu(w) \lesssim \int_{\mathbb{C}^n} \left|f(w)e^{-\frac{\alpha}{2}|w|^2}\right|^q \frac{\mu(D(w,r))}{(1+|w|)^s} dV(w) \tag{4.9}$$

for all entire function f in $\mathbb{C}^n$. upon setting $f = k_z$ and $q = t$ in it, we see that the left-hand side integral becomes $\widetilde{\mu}_{(t,s)}$ and

$$\begin{aligned}\widetilde{\mu}_{(t,s)}(z) &= \int_{\mathbb{C}^n} (1+|w|)^{-s}|k_z(w)|^t e^{-\frac{\alpha t}{2}|w|^2} d\mu(w)\\ &\lesssim \int_{\mathbb{C}^n} (1+|w|)^{-s}|k_z(w)|^t e^{-\frac{\alpha t}{2}|w|^2} \mu(D(w,r)) dV(w) = \widetilde{g}_{(t,s)}(z)\end{aligned}$$

where we set $g(w) = \mu(D(w,r))$. This coupled with Lemma 4.1 yield the reverse estimate in (4.7). That is

$$\|\widetilde{\mu}_{(t,s)}\|_{L^p} \lesssim \|\widetilde{g}_{(t,s)}\|_{L^p} \lesssim \|g\|_{L^p} = \|\mu_{(s,r,D)}\|_{L^p} \tag{4.10}$$

for all p. Since the case for $p=\infty$ is trivial, the proof will be complete once we show that the first and the last quantities in (4.5) are comparable for $1 \le p < \infty$. In doing so,

$$\begin{aligned}\int_{\mathbb{C}^n} \left(\frac{\mu(D(z,r))}{(1+|z|)^s}\right)^p dV(z) &\le \sum_{k=1}^{\infty} \int_{D(z_k,r)} \left(\frac{\mu(D(z,r))}{(1+|z|)^s}\right)^p dV(z)\\ &\le \sum_{k=1}^{\infty} \int_{D(z_k,r)} \left(\frac{\mu(D(z_k,2r))}{(1+|z_k|)^s}\right)^p dV(z)\\ &\lesssim \sum_{k=1}^{\infty} \left(\frac{\mu(D(z_k,2r))}{(1+|z_k|)^s}\right)^p.\end{aligned}$$

Here the last inequality follows since r is fixed and $Vl(D(z_k,r)) \simeq r^n$ independent of k. From this and Lemma 4.2 we obtained one side of the required estimate in (4.5), namely that

$$\|\mu_{(s,r,D)}\|_{L^p} \lesssim \|\mu_{(s,r,D)}(z_k)\|_{\ell^p}. \tag{4.11}$$

It remains to prove the reverse estimate in (4.11). To this end, Observe that

$$N_{\max} \int_{\mathbb{C}^n} \left(\frac{\mu(D(z,2r))}{1+|z|^s}\right)^p dV(z) \ge \sum_{k=1}^{\infty} \int_{D(z_k,r)} \left(\frac{\mu(D(z,2r))}{(1+|z|)^s}\right)^p dV(z).$$

Now for each $z \in D(z_k,r)$, we deduce from triangle inequality that $\mu(D(z,2r)) \ge \mu(D(z_k,r))$ and hence

$$\begin{aligned}\sum_{k=1}^{\infty} \int_{D(z_k,r)} \left(\frac{\mu(D(z,2r))}{(1+|z|)^s}\right)^p dV(z) &\ge \sum_{k=1}^{\infty} \int_{D(z_k,r)} \left(\frac{\mu(D(z_k,r))}{(1+|z_k|)^s}\right)^p dV(z)\\ &\gtrsim \sum_{k=1}^{\infty} \left(\frac{\mu(D(z_k,r))}{(1+|z_k|)^s}\right)^p\end{aligned}$$

from which and Lemma 4.2 again the required estimate follows. □

We remark that the norm estimates in Lemma 4.3 are also valid for $0<p<1$. Its proof requires a bite different approach than the one outlined above. We decided not to develop it here since we do not need such fact in our consideration.

Lemma 4.4. *Let $0 < q, p \le \infty$ and (g,ψ) be a pair of entire functions. Then $uC_\psi : \mathcal{F}^p_{(\alpha,m)} \to \mathcal{F}^q_{(\alpha,m)}$ is compact if and only if $\|uC_\psi f_k\|_{(q,m)} \to 0$ as $k \to \infty$ for each bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{F}^p_{(\alpha,m)}$ converging to zero uniformly on compact subsets of $\mathbb{C}^n$ as $k \to \infty$.*

The lemma can be proved following standard arguments; see also [26, Lemma 8]. The lemma will be used repeatedly in the proofs of our compactness results.

5. Proof of the main results

Proof of Theorem 2.1. The equivalencies of (ii), (iii), and (iv) come from Lemma 4.3. We now proceed to show that statements (iii) follows from (i) and (i) follows from (iv). Assume that μ is a (p,q) Fock–Carleson measure and consider a test function $K_w(z) = e^{\alpha\langle z,w\rangle}$ in $\mathcal{F}_{(m,p)}$. Note that this is the kernel function for $\mathcal{F}^2_{(0,\alpha)}$ at the point w. Then

$$\begin{aligned}\int_{\mathbb{C}^n} \frac{|K_w(z)|^q}{e^{\frac{\alpha q}{2}|z|^2}} d\mu(z) &\leq \|\mu\|^q \bigg(\int_{\mathbb{C}^n} \big|K_w(z) e^{-\frac{\alpha}{2}|z|^2}|z|^m\big|^p dV(z)\bigg)^{\frac{q}{p}}\\ &\lesssim \|\mu\|^q \bigg(\int_{\mathbb{C}^n} \big|K_w(z)e^{-\frac{\alpha}{2}|z|^2}(1+|z|)^m\big|^p dV(z)\bigg)^{\frac{q}{p}}. \qquad (5.1)\end{aligned}$$

By Lemma 20 of [9] or from a simple computation, the right-hand side integral is estimated as

$$\begin{aligned}\|\mu\|^q\bigg(\int_{\mathbb{C}^n}\frac{|K_w(z)|^p}{(1+|z|)^{-pm}}e^{-\frac{\alpha p}{2}|z|^2}dV(z)\bigg)^{\frac{q}{p}} &\lesssim \|\mu\|^q\bigg(\frac{e^{\frac{p\alpha}{2}|w|^2}}{(1+|w|)^{-mp}}\bigg)^{\frac{q}{p}}\\ &= \|\mu\|^q(1+|w|)^{mq}e^{\frac{q\alpha}{2}|w|^2}. \qquad (5.2)\end{aligned}$$

On the other hand, completing the square in the exponent on the left hand side of (5.1), we obtain

$$\begin{aligned}\int_{\mathbb{C}^n}\big|e^{\alpha\langle z,w\rangle-\frac{\alpha}{2}|z|^2}\big|^q d\mu(z) &= e^{\frac{q\alpha}{2}|w|^2}\int_{\mathbb{C}^n} e^{-\frac{q\alpha}{2}|z-w|^2}d\mu(z)\\ &\geq e^{\frac{q\alpha}{2}|w|^2}\int_{D(w,r)} e^{-\frac{q\alpha}{2}|z-w|^2}d\mu(z)\\ &\geq e^{\frac{q\alpha}{2}\left(|w|^2-r^2\right)}\mu(D(w,r)) \qquad (5.3)\end{aligned}$$

for all $w\in\mathbb{C}^n$. Combining this with (5.2) leads to (iii) and

$$\|\mu\|^q \gtrsim \big\|\mu_{(mq,r,D)}\big\|_{L^\infty}. \qquad (5.4)$$

We next show statement that (i) follows from (iv). The covering property of the sequence of balls $(D(z_k,r))_k$ implies

$$\int_{\mathbb{C}^n}|f(z)|^q e^{-\frac{\alpha q}{2}|z|^2}d\mu(z) \leq \sum_{k=1}^{\infty}\int_{D(z_k,2r)}|f(z)|^q e^{-\frac{\alpha q}{2}|z|^2}d\mu(z)$$

Using the estimate in (4.6), the sum is comparable to

$$\sum_{k=1}^{\infty}(1+|z_k|)^{-mq}\int_{D(z_k,2r)}|f(z)(1+|z|)^m|^q e^{-\frac{\alpha q}{2}|z|^2}d\mu(z)$$

which is bounded by

$$\sum_{k=1}^{\infty}\frac{\mu(D(z_k,2r))}{(1+|z_k|)^{mq}}\bigg(\sup_{z\in D(z_k,r))}|f(z)(1+|z|)^m|^p e^{-\frac{\alpha p}{2}|z|^2}\bigg)^{\frac{q}{p}}$$

$$\lesssim \sup_{k\geq 1}\frac{\mu(D(z_k,2r))}{(1+|z_k|)^{mq}}\bigg(\sum_{k=1}^{\infty}\int_{D(z_k,3r)}\frac{|f(z)(1+|z|)^m|^p}{e^{\frac{\alpha p}{2}|z|^2}}dV(z)\bigg)^{\frac{q}{p}} =: \mathcal{S}_1.$$

Now we claim that for each $f\in\mathcal{F}^p_{(m,\alpha)}$,

$$\sum_{k=1}^{\infty}\int_{D(z_k,3r)}\frac{|f(z)(1+|z|)^m|^p}{e^{\frac{\alpha p}{2}|z|^2}}dV(z)\lesssim\sum_{k=1}^{\infty}\int_{D(z_k,3r)}\frac{|f(z)|z|^m|^p}{e^{\frac{\alpha p}{2}|z|^2}}dV(z).$$

Because of (4.6) the claim trivially follows if $|z_k|\geq 1$ for all k. On the other hand, since (z_k) assumed to be a fixed $r/2$ lattice for $\mathbb{C}^n$, its covering property ensures that the inequality $|z_k|<1$ can happen for only a finite number of indices k. Thus there exists a positive constant M for which

$$\sum_{|z_k|<1}\int_{D(z_k,3r)}|f(z)|^p e^{-\frac{\alpha p}{2}|z|^2}dV(z)\leq M\sum_{|z_k|<1}\int_{D(z_k,3r)}\frac{|f(z)|^p|z|^{mp}}{e^{\frac{\alpha p}{2}|z|^2}}dV(z)$$

$$\lesssim\sum_{k=1}^{\infty}\int_{D(z_k,3r)}|f(z)|z|^m|^p e^{-\frac{\alpha p}{2}|z|^2}dV(z).$$

Observe that the analysis above in general implies

$$\int_{\mathbb{C}^n}|f(z)(1+|z|)^m|^p e^{-\frac{\alpha p}{2}|z|^2}dV(z)\simeq\int_{\mathbb{C}^n}|f(z)|z|^m|^p e^{-\frac{\alpha p}{2}|z|^2}dV(z). \quad (5.5)$$

Making use of this estimate, we obtain

$$\mathcal{S}_1\lesssim\sup_{k\geq 1}\frac{\mu(D(z_k,2r))}{(1+|z_k|)^{mq}}\|f\|^q_{(p,m)} \quad (5.6)$$

from which and Lemma 4.2, the statement in (i) and the estimate

$$\|\mu\|^q\lesssim\big\|(1+|z_k|)^{-mq}\mu(D(z_k,r))\big\|_{\ell^\infty}. \quad (5.7)$$

follow. To this end, the series of norm estimates in (2.2) follows from (4.5), (5.4) and (5.7).

Proof of Theorem 2.2. The equivalency of the statements in (ii), (iii) and (iv) follows easily from a simple modification of the proof of Lemma 4.1. Thus, we shall prove only (i) implies (iii) and (iv) implies (i). To prove the first, we consider a sequence of test functions $\xi_{(w,m)}$ defined by

$$\xi_{(w,m)}(z)=\frac{k_w(z)}{(1+|w|)^m}=\frac{e^{\alpha\langle z,w\rangle-\frac{\alpha}{2}|w|^2}}{(1+|w|)^m}$$

for each $w\in\mathbb{C}^n$. By Lemma 20 of [9] again, we have

$$\sup_{w\in\mathbb{C}^n}\|\xi_{(w,m)}\|_{(p,m)}<\infty$$

for all $p > 0$. It is also easily seen that $\xi_{(w,m)} \to 0$ as $|w| \to \infty$, and the convergence is uniform on compact subsets of $\mathbb{C}^n$. If μ is a (p,q) vanishing Fock–Carleson measure, then

$$\lim_{|w|\to\infty} \int_{\mathbb{C}^n} |\xi_{(w,m)}|^q e^{-\frac{\alpha q}{2}|z|^2} d\mu(z) = \lim_{|w|\to\infty} \int_{\mathbb{C}^n} \frac{e^{-\frac{\alpha q}{2}|z-w|^2}}{(1+|w|)^{mq}} d\mu(z) = 0,$$

from which we have

$$0 \geq \lim_{|w|\to\infty} \int_{D(w,r)} \frac{e^{-\frac{\alpha q}{2}|z-w|^2}}{(1+|w|)^{mq}} d\mu(z) \geq e^{-\frac{\alpha r^2}{2}} \lim_{|w|\to\infty} \frac{\mu(D(w,r))}{(1+|w|)^{mq}}.$$

Since the factor $e^{-\alpha r^2/2}$ is independent of w, the desired conclusion follows.

We now prove (iv) implies (i). Let f_j be a sequence in $\mathcal{F}^p_{(m,\alpha)}$ such that $\sup_j \|f_j\|_{(p,m)} < \infty$ and f_j converges to zero uniformly on each compact subset of $\mathbb{C}^n$ as $j \to \infty$. We aim to show that

$$\int_{\mathbb{C}^n} \left|f_j(z)e^{-\frac{\alpha}{2}|z|^2}\right|^q d\mu(z) \to 0$$

as $j \to \infty$. By hypothesis, for each $\epsilon > 0$, there exists a positive integer N_0 such that $\mu_{(mq,r,D)}(z_k) < \epsilon$ whenever $k \geq N_0$. Let U_o denotes the union of the closure of the balls $D(z_k, 3r), k = 1, ...N_0$. Then U_o is a compact subset of $\mathbb{C}^n$. Since f_j converges to zero uniformly on compact subsets of $\mathbb{C}^n$, there also exists $N_1 > N_0$ such that

$$\begin{aligned} Q_1 &= \sum_{k=1}^{N_0} \mu_{(mq,r,D)}(z_k) \bigg(\int_{D(z_k,3r)} (1+|z|)^{pm} |f_j(z)|^p e^{-\frac{\alpha p}{2}|z|^2} \bigg)^{\frac{q}{p}} \\ &\leq \epsilon \bigg(\sum_{k=1}^{N_0} \int_{D(z_k,3r)} |f_j(z)|^p \frac{(1+|z|)^{pm}}{e^{\frac{\alpha p}{2}|z|^2}} \bigg)^{\frac{q}{p}} \lesssim \epsilon \big(\sup_{z\in U_0} |f_j(z)|^p \big)^{\frac{q}{p}} \lesssim \epsilon \end{aligned} \tag{5.8}$$

for all $j \geq N_1$. On the other hand, for all $k \geq N_0$

$$\begin{aligned} Q_2 &= \sum_{N_0+1}^{\infty} \mu_{(mq,r,D)}(z_k) \bigg(\int_{D(z_k,3r)} (1+|z|)^{mp} |f_j(z)|^p e^{-\frac{\alpha p}{2}|z|^2} \bigg)^{\frac{q}{p}} \\ &\leq \epsilon \bigg(\sum_{N_0+1}^{\infty} \int_{D(z_k,3r)} (1+|z|)^{mp} |f_j(z)|^p e^{-\frac{\alpha p}{2}|z|^2} \bigg)^{\frac{q}{p}} \\ &\lesssim \epsilon \|f_j\|^q_{(p,m)} \lesssim \epsilon \end{aligned} \tag{5.9}$$

Thus, using (4.6), (5.8), (5.9), and sufficiently large $j \geq \max\{N_0, N_1\}$,

$$\begin{aligned} \int_{\mathbb{C}^n} \left|f_j(z)e^{-\frac{\alpha}{2}|z|^2}\right|^q d\mu(z) &\leq \sum_{k=1}^{\infty} \int_{D(z_k,r)} |f_j(z)|^q e^{-\frac{\alpha q}{2}|z|^2} d\mu(z) \\ &\lesssim \sum_{k=1}^{\infty} \mu_{(mq,r,D)}(z_k) \sup_{z\in D(z_k,r)} \frac{|f_j(z)|^q (1+|z|)^{mq}}{e^{\frac{\alpha q}{2}|z|^2}} \\ &\leq Q_1 + Q_2 \lesssim \epsilon. \end{aligned}$$

Proof of Theorem 2.3. Since $p/(p-q) \geq 1$, by Lemma 4.3, (iii), (iv) and (v) are again equivalent. To show that statement (v) follows from (i) we may follow the classical Leuecking's approach via Khinchine's equality in [19]. Consider a function f in $\mathcal{F}^p_{(m,\alpha)}$. Then by Lemma 1.1, $z^\beta f$ belongs to $\mathcal{F}^p_\alpha$ for all multi-indices β such that $\beta_1 + \beta_2 + ...\beta_n = m$. Thus, there exists a sequence $c_j \in \ell^p, 0 < p \leq \infty$, for which

$$z^\beta f(z) = \sum_{j=1}^\infty c_j k_{z_j}(z) \in \mathcal{F}^p_{(\alpha)} \text{ and } \quad \|f\|_{(p,m)} \simeq \||z|^m k_{z_j}\|_p \lesssim \|(c_j)\|_{\ell^p}. \tag{5.10}$$

This was proved in [17] for $p \geq 1$ and in [32] for $0 < p < 1$. We first assume that $0 < q < \infty$. Since μ is a (p,q) Fock–Carleson measure,

$$\int_{\mathbb{C}^n} |f(z)|^q e^{-\frac{\alpha q}{2}|z|^2} d\mu(z) \leq \|\mu\|^q \|f_c\|^q_{(p,m)} \lesssim \|\mu\|^q \|(c_j)\|^q_{\ell^p}.$$

If (r_j) is the Rademacher sequence of functions on $[0,1]$ chosen as in [19], then Khinchine's inequality yields

$$\left(\sum_{j=1}^\infty |c_j|^2 |k_{z_j}(z)|z|^{-m}|^2\right)^{q/2} \lesssim \int_0^1 \left|\sum_{j=1}^\infty c_j r_j k_{z_j}(z)|z|^{-m}\right|^q dt. \tag{5.11}$$

Note that here if the r_j are chosen as refereed above, then $(c_j r_j) \in \ell^p$ with $\|(c_j r_j)\|_{\ell^p} = \|(c_j)\|_{\ell^p}$ and

$$\sum_{j=1}^\infty c_j r_j k_{z_j}(z) z^{-\beta} \in \mathcal{F}^p_{(m,\alpha)}, \text{ with } \Big\|\sum_{j=1}^\infty c_j r_j(t) k_{(z_j,\alpha)}(z) z^{-\beta}\Big\|_{(p,m)} \lesssim \|(c_j)\|_{\ell^p}$$

for all multi-indices β such that $\beta_{sn} = m$. Making use of first (5.11) and subsequently Fubini's theorem, we obtain

$$\begin{aligned}
\int_{\mathbb{C}^n} &\left(|c_j|^2 |k_{z_j)}(z)|z|^{-m}|^2\right)^{q/2} d\mu(z) \\
&\lesssim \int_{\mathbb{C}^n} \left(\int_0^1 \left|\sum_{j=1}^\infty c_j r_j(t) k_{z_j}(z)|z|^{-m}\right|^q dt\right) d\mu(z) \\
&= \int_0^1 \left(\int_{\mathbb{C}^n} \left|\sum_{j=1}^\infty c_j r_j(t) k_{z_j}(z)|z|^{-m}\right|^q d\mu(z)\right) dt \\
&\lesssim \|\mu\|^q \|(c_j)\|^q_{\ell^p}.
\end{aligned} \tag{5.12}$$

Now if $q \geq 2$, then using (4.6), we have

$$\sum_{j=1}^\infty |c_j|^q \frac{\mu(D(z_j,r))}{(1+|z_j|)^{mq}} = \int_{\mathbb{C}^n} \sum_{j=1}^\infty |c_j|^q \frac{\chi_{D(z_j,r)}(z)}{(1+|z|)^{mq}} d\mu(z) \tag{5.13}$$

$$\leq \int_{\mathbb{C}^n} \left(\sum_{j=1}^\infty |c_j|^2 \frac{\chi_{D(z_j,r)}(z)}{(1+|z|)^{2m}}\right)^{q/2} d\mu(z) \tag{5.14}$$

where the last inequality is since $q/2 \geq 1$ and $|c_j| \geq 0$ for all j. On the other hand, if $q < 2$, then applying Hölder's inequality with exponent $2/q$ to the integral in (5.13) gives

$$\int_{\mathbb{C}^n} \sum_{j=1}^{\infty} |c_j|^q \frac{\chi_{D(z_j,r)}(z)}{(1+|z_j|)^{mq}} d\mu(z) \leq N_{\max}^{\frac{2-q}{2}} \int_{\mathbb{C}^n} \Bigg(\sum_{j=1}^{\infty} |c_j|^2 \frac{\chi_{D(z_j,r)}(z)}{(1+|z|)^{2m}} \Bigg)^{q/2} d\mu(z)$$
$$\lesssim \int_{\mathbb{C}^n} \Bigg(\sum_{j=1}^{\infty} |c_j|^2 \frac{\chi_{D(z_j,r)}(z)}{(1+|z|)^{2m}} \Bigg)^{q/2} d\mu(z).$$

The last integral here and in (5.14) are bounded by

$$e^{\frac{q\alpha}{2}r^2} \int_{\mathbb{C}^n} \Bigg(\sum_{j=1}^{\infty} |c_j|^2 \frac{e^{-\alpha|z-z_j|^2}}{(1+|z|)^{2m}} \Bigg)^{q/2} d\mu(z) \lesssim \int_{\mathbb{C}^n} \Big(|c_j|^2 |k_{z_j)}(z)|z|^{-m}|^2 \Big)^{q/2} d\mu(z).$$

This combined with (5.12) gives

$$\sum_{j=1}^{\infty} |c_j|^q \frac{\mu(D(z_j,r))}{(1+|z_j|)^{mq}} \lesssim \|\mu\|^q \|(c_j)\|_{\ell^p}^q = \|\mu\|^q \Big(\sum_{j=1}^{\infty} |c_j|^p \Big)^{q/p}. \tag{5.15}$$

Then a duality argument gives that

$$\mu_{(mq,r,D)}(z_j) \in \ell^{\frac{p}{p-q}} \quad \text{and} \quad \|\mu\|^q \gtrsim \|\mu_{(mq,r,D)}(z_j)\|_{\ell^{\frac{p}{p-q}}}. \tag{5.16}$$

We now prove (iv) implies (i). Integrating both side of (4.8) against the measure μ and subsequently using $\chi_{D(z,r)}(w) = \chi_{D(w,r)}(z)$, and Fubini's theorem we get

$$\int_{\mathbb{C}^n} |f(w)e^{-\frac{\alpha}{2}|w|^2}|^q d\mu(w) \lesssim \int_{\mathbb{C}^n} \big|f(w)e^{-\frac{\alpha}{2}|w|^2}\big|^q \mu(D(w,r)) dV(w)$$
$$= \int_{\mathbb{C}^n} \big|f(w)e^{-\frac{\alpha}{2}|w|^2}(1+|w|)^m\big|^q \mu_{(mq,r,D)} dV(w) \tag{5.17}$$

Applying Hölder's inequality with exponent p/q and (5.5)

$$\int_{\mathbb{C}^n} \big|f(w)e^{-\frac{\alpha}{2}|w|^2}(1+|w|)^m\big|^q \mu_{(mq,r,D) dV(w)} \lesssim \|f\|_{(p,m)}^q \big\|\mu_{(mq,r,D)}\big\|_{L^{\frac{p}{p-q}}}.$$

It follows from this that the estimate

$$\|\mu\|^q \lesssim \big\|\mu_{(mq,r,D)}\big\|_{L^{\frac{p}{p-q}}}$$

which together with (5.16) and (4.5) yields the series of norm estimates in (2.3).

Obviously, (ii) implies (i). We proceed to show its converse. Let f_j be a sequence of functions in $\mathcal{F}^p_{(m,\alpha)}$ such that $\sup_j \|f_j\|_{(p,m)} < \infty$ and f_j converges uniformly to zero on compact subsets of $\mathbb{C}^n$ as $j \to \infty$. For a fixed $R > \delta > 0$, we write

$$\int_{\mathbb{C}^n} |f_j(z)|^q e^{\frac{-q\alpha}{2}|z|^2} d\mu(z) = \Big(\int_{|z|\leq R-\delta} + \int_{|z|>R-\delta} \Big) |f_j(z)|^q e^{-\frac{q\alpha}{2}|z|^2} d\mu(z)$$
$$= I_{j1} + I_{j2}.$$

We estimate the two pieces of integrals independently and consider first I_{j1}. Since $f_j \to 0$ uniformly on compact subsets of $\mathbb{C}^n$ as $j \to \infty$, we find

$$\begin{aligned}\limsup_{j\to\infty} I_{j1} &= \limsup_{j\to\infty} \int_{|z|\leq R-\delta} |f_j(z)|^q e^{-\frac{q\alpha}{2}|z|^2} d\mu(z)\\ &\leq \limsup_{j\to\infty} \sup_{|z|\leq R-\delta} |f_j(z)|^q \int_{|z|\leq R-\delta} e^{-\frac{q\alpha}{2}|z|^2} d\mu(z)\\ &\lesssim \limsup_{j\to\infty} \sup_{|z|\leq R-\delta} |f_j(z)|^q \to 0, \text{ as } \ j\to\infty.\end{aligned}$$

If we denote by μ^R the truncation of μ on the set $\{z \in \mathbb{C}^n : |z| > R-\delta\}$, then applying (5.17) we obtain,

$$\begin{aligned}\limsup_{j\to\infty} I_{j2} &= \limsup_{j\to\infty} \int_{|z|>R-\delta} |f_j(z)|^q e^{-\frac{q\alpha}{2}|z|^2} d\mu(z)\\ &= \limsup_{j\to\infty} \int_{\mathbb{C}^n} |f_j(z)|^q e^{-\frac{q\alpha}{2}|z|^2} d\mu^R(z)\\ &\lesssim \limsup_{j\to\infty} \int_{\mathbb{C}^n} |f_j(z)|^q (1+|z|)^{mq} e^{-\frac{q\alpha}{2}|z|^2} \mu^R_{(mq,r,D)} dV(z).\end{aligned}$$

Applying Hölder's inequality again, we obtain

$$\begin{aligned}&\limsup_{j\to\infty} \int_{\mathbb{C}^n} |f_j(z)|^q (1+|z|)^{mq} e^{-\frac{q\alpha}{2}|z|^2} \mu^R_{(mq,r,D)} dV(z)\\ &\qquad\leq \limsup_{j\to\infty} \|f_j\|^q_{(p,m)} \int_{\mathbb{C}^n} \left|\mu^R_{(mq,r,D)}\right|^{\frac{p}{p-q}} dV(z)\\ &\qquad= \limsup_{j\to\infty} \|f_j\|^q_{(p,m)} \int_{|z|>R-r} \left|\mu^R_{(mq,r,D)}\right|^{\frac{p}{p-q}} dV(z).\end{aligned}$$

Since $\sup_j \|f_j\|_{(p,m)} < \infty$ and $\mu^R_{(mq,r,D)} \in L^{\frac{p}{p-q}}$, we let $R \to \infty$ in the above relation to conclude that μ is a (p,q) vanishing Fock–Carelson measure, and completes the proof of the theorem.

Proof of Theorem 2.4. The proof of the theorem closely follows the arguments used in the proof of Theorem 2.3. We will sketch only some of the required modifications below. The equivalencies of the statements in (iii), (iv) and (v) follow from Lemma 4.3 with $s = mq$. We observe that the global geometric condition (vi) follows from (iii) when we in particular set $t = 1$. Because by Fubini's theorem, we may have

$$\begin{aligned}\int_{\mathbb{C}^n} \widetilde{\mu}_{(1,mq)}(z) dV(z) &= \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} \frac{e^{\frac{\alpha}{2}|\langle w,z\rangle|^2-\frac{\alpha}{2}|z|^2-\frac{\alpha}{2}|w|^2}}{(1+|z|)^{mq}} d\mu(w) dV(z)\\ &= \int_{\mathbb{C}^n}\left(\int_{\mathbb{C}^n} \frac{e^{-\frac{\alpha}{2}|z-w|^2}}{(1+|z|)^{mq}} dV(z)\right) d\mu(w). \qquad (5.18)\end{aligned}$$

Since

$$(1+|z|)^{-1} \leq \frac{1+|z-w|}{1+|w|}, \ z, \ w \in \mathbb{C}^n,$$

the integral in (5.18) is bounded by

$$\int_{\mathbb{C}^n}(1+|w|)^{-mq}\int_{\mathbb{C}^n}(1+|z-w|)^{mq}\frac{e^{-\frac{\alpha}{2}|z-w|^2}}{(1+|z|)^{mq}}dV(z)d\mu(w)$$
$$\lesssim \int_{\mathbb{C}^n}\frac{1}{(1+|w|)^{mq}}d\mu(w)=\mu_{mq}(\mathbb{C}^n). \tag{5.19}$$

On the other hand, an application of (4.6) gives

$$\begin{aligned}\int_{\mathbb{C}^n}\bigg(\int_{\mathbb{C}^n}\frac{e^{-\frac{\alpha}{2}|z-w|^2}}{(1+|z|)^{mq}}dV(z)\bigg)d\mu(w) &\geq \int_{\mathbb{C}^n}\int_{D(w,1)}\frac{e^{-\frac{\alpha}{2}|z-w|^2}}{(1+|z|)^{mq}}dV(z)d\mu(w)\\ &\gtrsim e^{-\alpha/2}\int_{\mathbb{C}^n}\frac{1}{(1+|w|)^{mq}}d\mu(w)\\ &\simeq \mu_{mq}(\mathbb{C}^n).\end{aligned} \tag{5.20}$$

Combining (5.18), (5.19), and (5.20) we obtain

$$\int_{\mathbb{C}^n}\widetilde{\mu}_{(1,mq)}(z)dV(z)\simeq \mu_{mq}(\mathbb{C}^n). \tag{5.21}$$

This shows that shows that (vi) holds if and only if (iii) holds for $t=1$.

We now prove (i) implies (v). For this, we simply modify the proof of (i) implies (v) in the proof of Theorem 2.3. Thus, replace p by ∞ and follow the same arguments until we get equation (5.15) which would be in this case

$$\sum_{j=1}^{\infty}|c_j|^q\mu_{(mq,r,D)}(z_j)\lesssim \|\mu\|^q\|(c_j)\|_{\ell^\infty}^q. \tag{5.22}$$

Since (c_j) is an arbitrary sequence in ℓ^∞, we may in particular set $c_j=1$ for all j in the above relation to make the desired conclusion. Observe that this particular choice in (5.22) also ensures

$$\big\|\mu_{(mq,r,D)}(z_j)\big\|_{\ell^1}\lesssim \|\mu\|^q. \tag{5.23}$$

To prove that (i) follows from (iii), observe that applying (5.17) to a function f in $\mathcal{F}^\infty_{(m,\alpha)}$ gives

$$\begin{aligned}\int_{\mathbb{C}^n}\big|f(w)e^{-\frac{\alpha}{2}|w|^2}\big|^q d\mu(w) &\lesssim \int_{\mathbb{C}^n}\big|f(w)e^{-\frac{\alpha}{2}|w|^2}\big|^q(1+|w|)^{mq}\mu_{(mq,r,D)}(w)dV(w)\\ &\leq \|f\|_{(\infty,m)}^q\int_{\mathbb{C}^n}\mu_{(mq,r,D)}(w)dV(w)\\ &= \|f\|_{(\infty,m)}^q\|\mu_{(mp,r,D)}\|_{L^1}\end{aligned} \tag{5.24}$$

which completes the proof for (iii) implies (i). From (5.24), we also have

$$\|u\|\lesssim \|\mu_{(mp,r,D)}\|_{L^1}^{1/q} \tag{5.25}$$

from which, (5.23), (5.21) and (4.5), the series of norm estimates in (2.4) follow.

It remains to show (ii) follows from (i). But this can be easily done by simply modifying a similar proof in Theorem 2.3. Thus, we omit the details.

Proof of Theorems 3.1, 3.3 and 3.4. The central idea in these proofs is to translate the given problem into a (p,q) embedding map problem for the Fock–Sobolev spaces; through which we may invoke the notion of (p,q) Fock–Carleson measures and apply the results already proved in the preceding parts.

For each $p>0$, we set $\theta_{(m,p)}$ to be the positive pull back measure on $\mathbb{C}^n$ defined by

$$\theta_{(m,p)}(E)=\int_{\psi^{-1}(E)}|u(z)|^p|z|^{mp}e^{-\frac{\alpha p}{2}|z|^2}dV(z)$$

for every Borel subset E of $\mathbb{C}^n$. Then by substitution, we have

$$\begin{aligned}\|uC_{\psi)}f\|^q_{(m,q)}&\simeq\int_{\mathbb{C}^n}|f(z)|^q d\theta_{(m,q)}(z)=\int_{\mathbb{C}^n}|f(z)e^{-\frac{\alpha}{2}|z|^2}|^q e^{\frac{q\alpha}{2}|z|^2}d\theta_{(m,q)}(z)\\&=\int_{\mathbb{C}^n}|f(z)e^{-\frac{\alpha}{2}|z|^2}|^q d\lambda_{(m,q)}(z)\end{aligned}$$

where $d\lambda_{(m,q)}(z)=e^{\frac{q\alpha}{2}|z|^2}d\theta_{(m,q)}(z)$. This shows that $uC_\psi:\mathcal{F}^p_{(m,\alpha)}\to\mathcal{F}^q_{(m,\alpha)}$ is bounded if and only if $\lambda_{(m,q)}$ is a (p,q) Fock–Carleson measure. We may now consider three different cases depending on the size of the exponents.

Case 1: $p\leq q$**.** In this case, by Theorem 2.1, the boundedness of uC_ψ holds if and only if $\widetilde{\lambda}_{(m,q)}$ belongs to L^∞. But substituting back $d\lambda_{(m,q)}$ and $d\theta_{(m,q)}$ in terms of dV results in

$$\begin{aligned}\widetilde{\lambda}_{(m,q)}(z)&=\int_{\mathbb{C}^n}\frac{e^{-\frac{q\alpha}{2}|w-z|^2}}{(1+|w|)^{mq}}d\lambda_{(m,q)}(w)\\&=\int_{\mathbb{C}^n}\frac{|k_z(\psi(w))|^q e^{-\frac{q\alpha}{2}|w|^2}}{(1+|\psi(w)|)^{mq}}|u(w)|^q|w|^{mq}dV(w)\\&=B_{(m,\psi)}(|u|^q)(z).\end{aligned}$$

The norm estimate in (3.1) easily follows from the series of norm estimates in Theorem 2.1.

The proof of part (ii) of Theorem 3.1 is similar to the first part. This time we need to argue with Theorem 2.2 instead of Theorem 2.1. Thus, we omit the trivial details.

Case 2: $0<q<p<\infty$. By Theorem 2.3, $\lambda_{(m,q)}$ is a (p,q) Fock–Carleson measure if and only if $\lambda_{(m,q)}$ is a (p,q) vanishing Fock–Carleson measure. This again holds if and only if $\widetilde{\lambda}_{(m,q)}=B_{(m,\psi)}(|u|^q)$ belongs to $L^{p/(p-q)}$. The norm estimate in (3.7) also follows from the series of norm estimates in (2.3).

Case $3: 0<q<\infty$ **and** $p=\infty$. As in the previous cases, by Theorem 2.4, $\lambda_{(m,q)}$ is an (∞,q) Fock–Carleson measure if and only if $\lambda_{(m,q)}$ is an (∞,q) vanishing Fock–Carleson measure which is equivalent to the fact that $\widetilde{\lambda}_{(m,q)}=B_{(m,\psi)}(|u|^q)\in L^1$. The norm estimate in (3.8) follows again from the estimates in (2.4).

Proof of Theorem 3.5. We first note if $m = 0$, the function

$$B^{\infty}_{(m,\psi)}(|u|)(z) = \frac{|z|^m |u(z)|}{(1+|\psi(z)|)^m} e^{\frac{\alpha}{2}\left(|\psi(z)|^2 - |z|^2\right)} = |u(z)| e^{\frac{\alpha}{2}\left(|\psi(z)|^2 - |z|^2\right)},$$

and for this particular case, the theorem was proved in [26]. We now generalize the proof for any m. From a simple application of Lemma 3 of [9], we conclude

$$|f(z)| \leq \frac{\|f\|_{(p,m)}}{(1+|z|)^m} e^{\frac{\alpha}{2}|z|^2} \tag{5.26}$$

for each f in $\mathcal{F}^p_{(m,\alpha)}$ and $0 < p \leq \infty$. This implies

$$\begin{aligned} \|uC_\psi f\|_{(\infty,m)} &= \sup_{z\in\mathbb{C}^n} |u(z)||z|^m |f(\psi(z))| e^{\frac{-\alpha}{2}|z|^2} \\ &\leq \|f\|_{(p,m)} \sup_{z\in\mathbb{C}^n} \frac{|u(z)||z|^m}{(1+|\psi(z)|)^m} e^{\frac{\alpha}{2}|\psi(z)|^2 - \frac{\alpha}{2}|z|^2} \\ &= \|f\|_{(p,m)} \sup_{z\in\mathbb{C}^n} B^{\infty}_{(m,\psi)}(|u|)(z) \end{aligned}$$

from which one side of the estimate in (3.9),

$$\|uC_\psi\| \leq \|B^{\infty}_{(m,\psi)}(|u|)\|_{L^\infty}, \tag{5.27}$$

and the sufficiency of part (i) of the theorem follow.

To prove the necessity part of the theorem, for each point $w \in \mathbb{C}^n$ we use again the sequence of test functions $\xi_{(w,m)}(z) = (1+|w|)^{-m} k_w(z)$. Then

$$\|\xi_{(w,m)}\|_{(p,m)} \lesssim 1 \tag{5.28}$$

independent of p and w which follows by Lemma 20 of [9] for $p < \infty$ and from a simple argument for $p = \infty$. Applying uC_ψ to $\xi_{(w,m)}$ and completing the square on the exponent yields

$$\|uC_\psi\| \gtrsim \|uC_\psi \xi_{(w,m)}\|_{(\infty,m)} \geq \frac{|u(z)||z|^m}{(1+|w|)^m} e^{\frac{\alpha}{2}\left(|\psi(z)|^2 - |\psi(z)-w|^2 - |z|^2\right)}$$

for all points w and z in $\mathbb{C}^n$. Setting $w = \psi(z)$ in particular leads to

$$\|uC_\psi\| \gtrsim \frac{|u(z)||z|^m}{(1+|\psi(z)|)^m} e^{\frac{\alpha}{2}|\psi(z)|^2 - \frac{\alpha}{2}|z|^2} = B^{\infty}_{(m,\psi)}(|u|)(z)$$

from which the necessity of the condition and the remaining side of the estimate in (3.9) follow.

To prove the second part of the theorem, we first assume that uC_ψ is compact. The sequence $\xi_{(w,m)}$ converges to zero as $|w| \to \infty$, and the convergence is uniform on compact subset of $\mathbb{C}^n$. We further assume that there exists sequence of points $z_j \in \mathbb{C}^n$ such that $|\psi(z_j)| \to \infty$ as $j \to \infty$. If such a sequence does not exist, then (3.10) holds trivially. It follows from compactness of uC_ψ that

$$\limsup_{j\to\infty} B^{\infty}_{(m,\psi)}(|u|)(z_j) \leq \limsup_{j\to\infty} \|uC_\psi \xi_{(\psi(z_j)),m}\|_{(\infty,m)} = 0 \tag{5.29}$$

from which (3.10) follows.

We next suppose that uC_ψ is bounded and condition (3.10)holds. We proceed to show compactness of uC_ψ. The condition along with Theorem 3.1 implies that uC_ψ is a bounded map. On the other hand, the function $f(z) = 1$ belongs to $\mathcal{F}_{(p,m)}$, in deed, a computation along (1.2) results in $\|f\|_{(p,m)} = 1$. It follows that by boundedness, the weight function u belongs to $\mathcal{F}^\infty_{(m,\alpha)}$. Let f_j be a sequence of functions in $\mathcal{F}^p_{(m,\alpha)}$ such that $\sup_j \|f_j\|_{(p,m)} < \infty$ and f_j converges uniformly to zero on compact subsets of $\mathbb{C}^n$ as $j \to \infty$. For each $\epsilon > 0$ by (3.10) there exists a positive N_1 such that

$$B^\infty_{(m,\psi)}(|u|)(z) < \epsilon$$

for all $|\psi(z)| > N_1$. From this together and (5.26), we obtain

$$\begin{aligned} |uC_\psi f_j(z)||z|^m e^{-\frac{\alpha}{2}|z|^2} &= |u(z)f_j(\psi(z))||z|^m e^{-\frac{\alpha}{2}|z|^2} \\ &\le \|f_j\|_{(p,m)} \frac{|u(z)||z|^m}{(1+|\psi(z)|)^m} e^{\frac{\alpha}{2}|\psi(z)|^2 - \frac{\alpha}{2}|z|^2} \lesssim \epsilon \end{aligned}$$

for all $|\psi(z)| > N_1$ and all j. On the other hand if $|\psi(z)| \le N_1$, then it easily seen that

$$\begin{aligned} |u(z)f_j(\psi(z))||z|^m e^{-\frac{\alpha}{2}|z|^2} &\le \|u\|_{(\infty,m)} \sup_{z:|\psi(z)|\le N_1} |f_j(\psi(z))| \\ &\lesssim \sup_{z:|\psi(z)|\le N_1} |f_j(\psi(z))| \to 0 \end{aligned}$$

as $j \to \infty$, and completes the proof.

Proof of Corollary 3.6. We first assume that $uC_\psi : \mathcal{F}^p_{(m,\alpha)}(\text{ or } \mathcal{F}^\infty_{(0,m,\alpha)}) \to \mathcal{F}^\infty_{(0,m,\alpha)}$ is compact and aim to verify condition (3.11). It follows that $uC_\psi : \mathcal{F}^p_{(m,\alpha)}(\text{ or } \mathcal{F}^\infty_{(0,m,\alpha)}) \to \mathcal{F}^\infty_{(m,\alpha)}$ is also compact. Then by part (ii) of Theorem 3.5, for each ϵ, there exists a positive integer N_1 such that

$$B^\infty_{(m,\psi)}(|u|)(z) < \epsilon$$

for all $|\psi(z)| > N_1$. On the other hand, setting $f(z) = 1$, by boundedness (which follows from compactness) we have $u \in \mathcal{F}^\infty_{(0,m,\alpha)}$. Thus, there exists a positive integer N_2 for which

$$|u(z)||z|^m e^{-\frac{\alpha}{2}|z|^2} < \epsilon e^{-\frac{\alpha}{2}N_1^2}$$

for all $|z| > N_2$. Therefore, if $|z| > N_2 > N_1$ we have

$$B^\infty_{(m,\psi)}(|u|)(z) \le e^{\frac{\alpha N_1^2}{2}} |u(z)|z|^m e^{\frac{-\alpha}{2}|z|^2} < \epsilon$$

as desired.

For the converse, let f_j be a uniformly bounded sequence of functions in $\mathcal{F}^p_{(m,\alpha)}(\text{ or } \mathcal{F}^\infty_{(0,m,\alpha)})$ which converges uniformly to zero on compact subsets of $\mathbb{C}^n$ as $j \to \infty$. For each $\epsilon > 0$, condition (3.11) implies that there exists a positive integer N_3 for which $B^\infty_{(m,\psi)}(|u|)(z) < \epsilon$ for all $|z| > N_3$. From this and (5.26), we obtain

$$|uC_\psi f_j(z)||z|^m e^{-\frac{\alpha}{2}|z|^2} \le \sup_{j\ge 1} \|f_j\|_{(p,m)} \frac{|u(z)||z|^m}{(1+|\psi(z)|)^m} e^{\frac{\alpha}{2}|\psi(z)|^2 - \frac{\alpha}{2}|z|^2} \lesssim \epsilon$$

for all $|z| > N_3$ and all exponents p. On the other hand, since the set $\{|\psi(z)| : |z| \leq N_3\}$ is compact, there exists a positive integer N_4 for which $\sup_{|z|\leq N_3} |\psi(z)| \leq N_4$. Thus,

$$\begin{aligned} |u(z)f_j(\psi(z))||z|^m e^{-\frac{\alpha}{2}|z|^2} &\leq \|u\|_{(\infty,m)} \sup_{|w|\leq N_4} |f_j(w)| \\ &\lesssim \sup_{|w|\leq N_4} |f_j(w)| \to 0, \ j \to \infty. \end{aligned}$$

Proof of Theorem 3.7. The proof of the theorem follows a classical approach used to prove similar results in [13, 21, 26, 28, 29, 30]. Recall that each entire function f can be expressed as $f(z) = \sum_{k=0}^{\infty} p_k(z)$ where the function p_k are polynomials of degree k. We consider a sequence of operators R_j defined by

$$(R_j f)(z) = \sum_{k=j}^{\infty} p_k(z). \tag{5.30}$$

It was proved in [14, 28] that

$$\lim_{j\to\infty} \|R_j f\|_p = 0 \tag{5.31}$$

for each f in the ordinary Fock spaces $\mathcal{F}_\alpha^p$, and $1 < p < \infty$. Thus by the uniform boundedness principle

$$\sup_{j\geq 1} \|R_j\| < \infty. \tag{5.32}$$

Now if $h \in \mathcal{F}^p_{(m,\alpha)}$, then by Lemma 1.1 and (5.31)

$$\lim_{j\to\infty} \|R_j h\|_{(p,m)} \simeq \lim_{j\to\infty} \|R_j(z^\beta h)\|_p = 0 \tag{5.33}$$

from which the same conclusion (5.32) follows when the sequence (R_n) is defined on weighted Fock–Sobolev space.

Let $1 < p \leq q \leq \infty$ and assume that $uC_\psi : \mathcal{F}^p_{(m,\alpha)} \to \mathcal{F}^q_{(m,\alpha)}$ is bounded. Then mimicking the proof of Lemma 2 in [30] yields

$$\|uC_\psi\|_e \leq \liminf_{j\to\infty} \|uC_\psi R_j\|_{(q,m)}. \tag{5.34}$$

Having singled out this important inequality, we now proceed to prove the lower estimates in the theorem. To this end, let Q be a compact operator acting between $\mathcal{F}^p_{(m,\alpha)}$ and $\mathcal{F}^q_{(m,\alpha)}$. We first suppose that $q = \infty$. Since $\xi_{(w,m)}$ converges to zero uniformly on compact subset of $\mathbb{C}^n$ as $|w| \to \infty$ and (5.28) holds, we have

$$\begin{aligned} \|uC_\psi - Q\| &\geq \limsup_{|w|\to\infty} \|uC_\psi \xi_{(w,m)} - Q\xi_{(w,m)}\|_{(\infty,m)} \\ &\geq \limsup_{|w|\to\infty} \|uC_\psi \xi_{(w,m)}\|_{(\infty,m)} - \|Q\xi_{(w,m)}\|_{(\infty,m)} \\ &= \limsup_{|w|\to\infty} \|uC_\psi \xi_{(w,m)}\|_{(\infty,m)} \\ &\geq \limsup_{|\psi(w)|\to\infty} B^\infty_{(\psi,m)}(|u|)(z)(w), \end{aligned} \tag{5.35}$$

where the first equality is due to compactness of Q.

For $0 < q < \infty$, we consider a different sequence of test functions in $\mathcal{F}^P_{(m,\alpha)}$, namely that$k_w$; the normalized reproducing kernel function in $\mathcal{F}^2_{(0,\alpha)}$. This sequence replaces the role played by $\xi_{(w,m)}$ above and running the same procedure as in (5.35) gives

$$\begin{aligned}
\|uC_\psi - Q\| &\geq \limsup_{|w|\to\infty} \|uC_\psi k_w\|_{(q,m)} - \|Q\varphi_w\|_{(q,m)}\\
&= \limsup_{|w|\to\infty} \|uC_\psi k_w\|_{(q,m)}\\
&\geq \limsup_{|w|\to\infty} \int_{\mathbb{C}^n} \frac{|u(z)|^q|z|^{mq}}{(1+|\psi|)^{mq}} |k_w(\psi(z))|^q e^{-\frac{\alpha q}{2}|z|^2} dV\\
&= \Big(\limsup_{|w|\to\infty} B_{(\psi,m)}(|u|^q)(w)\Big)^{\frac{1}{q}}.
\end{aligned}$$

From this and (5.35) the lower estimate in (3.12) follows.

To prove the upper estimate, we again consider the next two different cases.

Case 1: Suppose $q < \infty$. Then for each f of unit norm in $\mathcal{F}^p_{(m,\alpha)}$, we get

$$\begin{aligned}
\|uC_\psi R_j f\|^q_{(m,q)} &\simeq \int_{\mathbb{C}^n} |R_j f(z)|^q d\theta_{(m,q)}(z)\\
&= \Big(\int_{\mathbb{C}^n\setminus D(0,\delta)} + \int_{D(0,\delta)}\Big)|R_j f(z)|^q e^{-\frac{\alpha q}{2}|z|^2} d\lambda_{(m,q)}(z) \qquad (5.36)
\end{aligned}$$

where again $d\lambda_{(m,q)}(z) = e^{\frac{q\alpha}{2}|z|^2} d\theta_{(m,q)}(z)$ and for some fixed $\delta > 0$. By Theorem 3.1, the first integral in (5.36) is bounded by

$$\|R_j f\|^q_{(p,m)}\Big(\sup_{z\in\mathbb{C}^n\setminus D(0,\delta)} B_{(\psi,m)}(|u|^q)(z)\Big) \lesssim \sup_{z\in\mathbb{C}^n\setminus D(0,\delta)} B_{(\psi,m)}(|u|^q)(z)$$

where we used the fact that $\sup_j \|R_j\| < \infty$. It remains to estimate the second integral in (5.36). Again by Theorem 3.1 and followed by the n-variable version of Lemma 3 in [30], the integral is estimated as

$$\begin{aligned}
&\int_{D(0,\delta)} |R_j f(z)|^q e^{-\frac{\alpha q}{2}|z|^2} d\lambda_{(m,q)}(z)\\
&\qquad\lesssim \sup_{z\in\mathbb{C}^n} B_{(\psi,m)}(|u|^q)(z) \int_{D(0,\delta)} |z|^{mp}|R_j f(z)|^p e^{-\frac{\alpha p}{2}|z|^2} dV(z)\\
&\qquad\lesssim \sup_{z\in\mathbb{C}^n} B_{(\psi,m)}(|u|^q)(z) I_j \int_{\mathbb{C}^n} e^{-\frac{p\alpha}{2}|z|^2} dV(z)\\
&\qquad\lesssim \sup_{z\in\mathbb{C}^n} B_{(\psi,m)}(|u|^q)(z) I_j
\end{aligned}$$

where

$$I_j \simeq \bigg(\sum_{m=j}^{\infty} (\delta\alpha)^m \sum_{\beta_{ns}=m} (\beta!)^{-1} \prod_{l=1}^{n} \Big(\frac{2}{\alpha s}\Big)^{\frac{\beta_l}{2}+\frac{1}{s}} \Big(\Gamma\Big(\frac{s\beta_l}{2}+1\Big)\Big)^{\frac{1}{s}}\bigg)^p \qquad (5.37)$$

with s the conjugate exponent of p and $\beta! = \prod_{l=1}^n \beta_l!$. Observe that by Stirling's approximation formula, we have

$$\Big(\Gamma\Big(\frac{s\beta_l}{2}+1\Big)\Big)^{\frac{1}{s}} \simeq \Big(\frac{\beta_l s}{2}\Big)^{\frac{\beta_l}{2}+\frac{1}{s}-\frac{1}{2s}} e^{-\frac{\beta_l}{2}}. \tag{5.38}$$

Plugging this in (5.37) and applying the ration test it is easily seen that the series converges and hence $I_j \to 0$ as $j \to \infty$. Thus, the contribution from the second integral in (5.36) goes to zero for large enough j. Therefore

$$\lim_{j\to\infty} \sup_{\|f\|_{(p,m)}=1} \|uC_\psi R_j f\|_{(q,m)}^q \lesssim \sup_{z\in\mathbb{C}^n\setminus D(0,\delta)} B_{(\psi,m)}(|g|^q)(z).$$

By (5.34) we get

$$\|uC_\psi\|_e^q \lesssim \lim_{\delta\to\infty} \sup_{z\in\mathbb{C}^n\setminus D(0,\delta)} B_{(\psi,m)}(|u|^q)(z) \simeq \limsup_{|z|\to\infty} B_{(\psi,m)}(|u|^q)(z)$$

and completes the proof for the first case.

Case 2: $q = \infty$. Not much effort is needed to prove this case since it follows by a simple modification of the arguments used in the previous case. We shall sketch it out for simplicity of the exposition. Acting similarly as above, for each f of unit norm in $\mathcal{F}^p_{(m,\alpha)}$, we may invoke (5.26) to get

$$\begin{aligned} |z|^m|uC_\psi R_j f(z)|e^{-\frac{\alpha}{2}|z|^2} &= |u(z)||z|^m|R_j f(\psi(z))|e^{\frac{-\alpha}{2}|z|^2} \\ &\leq \|R_j f\|_{(p,m)} \frac{|u(z)||z|^m}{(1+|\psi(z)|)^m} e^{\frac{\alpha}{2}|\psi(z)|^2-\frac{\alpha}{2}|z|^2} \\ &\leq \sup_{j\geq 1}\|R_j\| B^\infty_{(m,\psi)}(|u|)(z) \lesssim B^\infty_{(m,\psi)}(|u|)(z) \end{aligned}$$

from which we have that

$$\sup_{|\psi(z)|\geq\delta} |z|^m|uC_\psi R_j f(z)|e^{-\frac{\alpha}{2}|z|^2} \lesssim \sup_{|\psi(z)|\geq\delta} B^\infty_{(m,\psi)}(|u|)(z).$$

On the other hand, since uC_ψ is bounded, the weight function u belongs to $\mathcal{F}^\infty_{(m,\alpha)}$. Thus by Lemma 3 in [30] again, we have

$$\sup_{|\psi(z)|<\delta} |z|^m|uC_\psi R_n f(z)e^{-\frac{\alpha}{2}|z|^2}| \leq \|u\|_{(\infty,m)} I_j$$

where I_j and s are as in (5.37). But it is again easily seen that $I_j \to 0$ as $j \to \infty$. Therefore,

$$\liminf_{j\to\infty} \sup_{\|f\|_{(p,m)}=1} \sup_{|\psi(z)|\leq\delta} |uC_\psi R_j f(z)||z|^m e^{-\frac{\alpha}{2}|z|^2} = 0$$

and hence

$$\|uC_\psi\|_e \lesssim \sup_{|\psi(z)|\geq\delta} B^\infty_{(m,\psi)}(|u|)(z)$$

from which we get

$$\|uC_\psi\|_e \lesssim \limsup_{|\psi(z)|\to\infty} B^\infty_{(m,\psi)}(|u|)(z)$$

after letting δ to ∞, and completes the proof.

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